Efficiency Intervals, Rank Intervals and Dominance Relations of Decision-Making Units with Fixed-Sum Outputs

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Abstract:

How to evaluate the performance of decision-making units (DMUs) with fixed-sum outputs is a challenging question in data envelopment analysis (DEA). Recently, this challenge has been addressed by determining the common equilibrium efficient frontier; but because there can be several feasible equilibrium efficient frontiers, it can be difficult to provide valid results. To address this problem, we consider all feasible equilibrium efficient frontiers and develop several models to obtain the corresponding efficiency intervals, rank intervals as well as dominance relations for DMUs with fixed-sum outputs. We illustrate the proposed approach with two numerical examples and show that it gives more informative results than earlier DEA approaches.

Keywords:

Data envelopment analysis (DEA); Fixed-sum outputs; Efficiency interval; Rank interval; Dominance relations

1. Introduction

Data envelopment analysis (DEA) developed by Charnes et al. (1978) is a nonparametric approach to measure the performance of peer decision-making units (DMUs) with multiple inputs and outputs. Because DEA models such as CCR (Charnes et al., 1978) and BCC (Banker et al., 1984) are simple and can be readily solved, DEA has been employed in many areas, including efficiency analyses of Olympic Games (Lozano et al., 2002; Li et al., 2008; Lei et al., 2015), banks (Kao and Liu, 2004; Camanho and Dyson, 2005; Alireza et al., 2017), environments (Dai et al., 2014; Lozano and Sebastian, 2017; Burdon and Li, 2019) and hospitals (Zhang et al., 2011; Du et al., 2014), among others.

However, Dyson et al. (2001) pointed out several pitfalls in applying DEA. One of them is called "exogenous and constrained factors"[1], referring to situations in which some factors controlled by DMUs are constrained by the scale used, such as percentages. For example, the market share commonly used as an output in performance evaluation is fixed (Yang et al., 2014), because the total is 100%. Analogously, in the performance evaluation of nations in Olympic Games, the sum of gold (silver or bronze) medals is fixed so that if one country gets more of them, others get less (Lins et al., 2003; Wu et al., 2009; Du et al., 2018). In this paper, we call these kinds of factors or outputs as *fixed-sum outputs*. Dyson et al. (2001) suggested two approaches for addressing them: (1) using input-oriented DEA models and (2) developing new DEA approaches that satisfy fixed-sum constraints of outputs.

The first approach does not seem suitable when using inputs such as gross domestic product (GDP) and population, for instance. In the input-oriented DEA model, inefficient DMUs would need to improve their efficiencies by reducing their use of inputs; but the aim of reducing GDP or population would be ridiculous (Lins et al., 2003; Li et al., 2008). Consequently, several authors approach this pitfall based on the second approach. For example, Lins et al. (2003) were the first to propose a zero sum gains (ZSG) DEA approach which they applied to evaluate the performance of

[1]For details, readers can refer to Pitfall 5.4 in Dyson et al. (2001).

countries in the 2000 Sydney Olympic Games. Zero sum means that the sum of outputs' (gold, silver or bronze medals) adjustments across all participating nations is zero. Lins et al. (2003) propose two adjustment strategies, the equal reduction strategy and the proportional reduction strategy, whereby adjustments in first strategy can lead to negative outputs while those in the second cannot. Because the ZSG-DEA model with the second adjustment strategy is nonlinear, Bi et al. (2013) transformed it into a linear one. The resulting ZSG-DEA model has been applied to measure the performance of DMUs producing undesired outputs (Gomes and Lins 2008; Chiu et al. 2013). It has also been extended to a non-radial ZSG-DEA model (Fonseca et al. 2010) as well as to a fixed-sum outputs DEA (FSODEA) model based on minimizing output adjustments across all DMUs (Yang et al. 2011). Wu et al. (2014) apply the FSODEA model to measure environmental efficiency with undesirable fixed-sum outputs.

Yang et al. (2014) found that the novel ZSG-DEA model and its extensions such as the FSODEA model evaluate DMUs based on different efficient frontiers. This means that DMUs are, in effect, evaluated based on different evaluation platforms, making it hard to compare the efficiency results from these models. Therefore, Yang et al. (2014) proposed an equilibrium efficient frontier data envelopment analysis (EEFDEA) approach in which all DMUs are evaluated based on a common efficient frontier; but this approach can be very demanding computationally, because determining the efficient equilibrium frontier may require a large number of steps especially when the number of DMUs is large. Yang et al. (2015) therefore proposed a new approach which can generate a common equilibrium efficient frontier in a single step. However, the equilibrium efficient frontier based on Yang et al. (2014) and Yang et al. (2015) may not be unique. Fang (2016) attempted to achieve a unique equilibrium efficient frontier via a secondary goal approach. Zhu et al. (2017) showed that this approach cannot ensure that the equilibrium efficient frontier is unique and consequently sought to improve it through additional constraints such as assurance regions (AR-I type).

While the above approaches are helpful in choosing a unique equilibrium efficient frontier, they have two main drawbacks. First, in many applications there are no clear principles for how these additional constraints should be generated and, even if such constraints are introduced, it is difficult to test the uniqueness of the resulting equilibrium efficient frontier. Second, these approaches evaluate DMUs just based on a single feasible equilibrium efficient frontier, neglecting other feasible frontiers and their implications for efficiency analysis.

These observation motivate a new research question: Is it possible to evaluate all DMUs based on *all* feasible equilibrium efficient frontiers? This question is challenging because there can be an infinite number of feasible equilibrium efficient frontiers so that it is impossible to enumerate them all. Furthermore, because the equilibrium efficient frontier based on the generalized equilibrium efficient frontier data envelopment analysis (GEEFDEA) approach may be not unique, results concerning the efficiency of DMU may vary greatly depending on which one of the equilibrium efficient frontiers is selected. In this paper, we develop several models to address this question and apply them to obtain robust efficiency results, including rank intervals as well as dominance relations among all DMUs.

The structure of this paper is as follows. Section 2 reviews the GEEFDEA models and Section 3 presents models for establishing ranking interval and dominance relations. Section 4 illustrates the proposed efficiency measures based on a small dataset from the literature. Section 5 applies the proposed models to measure the performance of appliance industry companies. The last section concludes with suggestions for future research directions.

2. GEEFDEA Models

We first introduce the GEEFDEA approach proposed by Yang et al. (2015) for finding a common equilibrium efficient frontier for the evaluation of all DMUs. This approach consists of two models. One generates a common equilibrium efficient frontier and the other evaluates each DMU based on this frontier.

Assume that there are n DMUs and each DMU consumes m inputs x_{ij}, $i = 1,\ldots,m$, $j = 1,\ldots,n$ to produce s variant-sum outputs y_{rj}, $r = 1,2,\ldots,s$ and l fixed-sum outputs f_{tj}, $t = 1,\ldots,l$. The variant-sum outputs are freely disposable while the fixed-sum outputs satisfy the constraints $\sum_{j=1}^{n} f_{tj} = F_t, t = 1, \dots, l$ where F_t is a constant.

The first model in the GEEFDEA approach is

$$min \sum_{j=1}^{n} \sum_{t=1}^{l} w_t \alpha_{tj}$$

$$s.t. \frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{l} w_t (f_{tj} + \delta_{tj})}{\sum_{i=1}^{m} v_i x_{ij}} = 1 \quad \forall j$$

$$\sum_{j=1}^{n} \delta_{tj} = 0 \quad \forall t \qquad (1)$$

$$\alpha_{tj} = max\{\delta_{tj}, 0\}, \quad \forall t, j$$

$$f_{tj} + \delta_{tj} \geq 0, \quad \forall t, j$$

$$u_r, v_i, w_t \geq 0, \delta_{tj}\ free,$$

where u_r, w_t and v_i represent weights of the variant-sum output y_r, the fixed-sum output f_t and the input x_i, respectively. The term δ_{tj} denotes the t-th output adjustment of DMU$_j$ and it can be positive, negative or zero. Model (1) can be transformed to the following

$$min \sum_{j=1}^{n} \sum_{t=1}^{l} w_t |\delta_{tj}|$$

$$s.t. \frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{l} w_t (f_{tj} + \delta_{tj})}{\sum_{i=1}^{m} v_i x_{ij}} = 1 \quad \forall j$$

$$\sum_{j=1}^{n} \delta_{tj} = 0 \quad \forall t \qquad (2)$$

$$f_{tj} + \delta_{tj} \geq 0, \quad \forall t, j$$

$$u_r, v_i, w_t \geq 0, \delta_{tj}\ free.$$

This nonlinear model can be transformed into a linear one with the algorithm in Yang et al. (2015). Models (1) and (2) characterize the equilibrium efficient frontiers in one step. For details, readers are referred to Appendix A.

Suppose the optimal solution of δ_{tj} in model (2) is $\delta_{tj}^*, t = 1, \dots l, j = 1, \dots, n$. Then each DMU$_j$ adjusts its fixed-sum output from f_{tj} to $f_{tj} + \delta_{tj}^*$, $t = 1, \dots, l$

while keeping its inputs and variant-sum outputs unchanged. These adjusted DMUs have a common equilibrium efficient frontier, because they are all efficient by the first set of constraints in Model (2).

Based on the common equilibrium efficient frontier, Yang et al. (2015) evaluate performance of DMUs by solving the optimization problem

$$e_k^{GEEFDEA} = Min \frac{\sum_{i=1}^{m} v_i x_{ik}}{\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk}}$$

$$s.t. \ \frac{\sum_{i=1}^{m} v_i x_{ij}}{\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{l} w_t \left(f_{tj} + \delta_{tj}^{*}\right)} \geq 1 \quad \forall j \qquad (3)$$

$$u_r, v_i, w_t \geq 0, \quad \forall r, t, i,$$

where $\delta_{tj}^{*}, \forall t, j$ is the optimal value of δ_{tj} in Model (2). These constraints ensure that all DMUs are evaluated based on a common equilibrium efficient frontier.

However, the equilibrium efficient frontier based on Yang et al. (2015) may not be unique (Fang, 2016; Zhu et al., 2017). To see this, suppose that the optimal value of the objective function in Model (2) is $opti^*$. Then, all equilibrium efficient frontiers are characterised by the equations

$$\sum_{j=1}^{n} \sum_{t=1}^{l} w_t \left|\delta_{tj}\right| = opti^*$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{l} w_t (f_{tj} + \delta_{tj})}{\sum_{i=1}^{m} v_i x_{ij}} = 1 \quad \forall j$$

$$\sum_{j=1}^{n} \delta_{tj} = 0 \quad \forall t \qquad (4)$$

$$f_{tj} + \delta_{tj} \geq 0 \quad \forall t, j$$

$$u_r, v_i, w_t \geq 0, \delta_{tj} \ free.$$

In Model (4), there are $m+s+l+n*l$ variables to be determined, but only $n+l$ equality constraints. Therefore, there is considerable flexibility in choosing $\delta_{tj}^{*}, \forall t, j$ and consequently there can exist multiple equilibrium efficient frontiers. Zhu et al. (2017) propose a series of models for calculating the minimal and maximal $\delta_{tj}^{*}, \forall t, j$ for

DMU$_j$. Their numerical examples show there are multiple solutions $\delta_{tj}^{*}, \forall t, j$ as well as multiple equilibrium efficient frontiers.

For the convenience, we introduce the notations $u = (u_1, \ldots, u_s)^T$, $w = (w_1, \ldots, w_l)^T$, $v = (v_1, \ldots, v_m)^T$, $\delta = (\delta_{tj}), \forall t, j$. Then, the feasible set of solutions for Model (4) can be represented as

$$S = \{(\mathrm{u}, \mathrm{v}, \mathrm{w}, \delta) | \mathrm{u}, \mathrm{v}, \mathrm{w}, \delta \text{ satisfy Model (4)}\} \quad (R1)$$

Because this set *S* contains all feasible equilibrium efficient frontiers, we can evaluate all DMUs by calculating their efficiency intervals, rank intervals and dominance relations over the set *S* regardless of how many feasible equilibrium efficient frontiers there are.

3. Models for Efficiency, Rank Intervals and Dominance Relations

3.1. Efficiency Intervals

According to Podinovski (2001), the efficiency of DMU$_k$ can be defined as

$$E_k(u, v, w, \delta) = \frac{\sum_r u_r y_{rk} + \sum_t w_t f_{tk}}{\sum_i v_i x_{ik}} \quad \text{and} \quad (\mathrm{u}, \mathrm{v}, \mathrm{w}, \delta) \in S. \qquad \text{(R2)}$$

Because there can be multiple equilibrium efficient frontiers, the efficiency of each DMU based on Model (3) can vary depending on which frontier is being considered. Here, we adjust Model (R2) to calculate the minimal and maximal efficiency values of each DMU as

$$max/min \frac{\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk}}{\sum_i v_i x_{ik}}$$

$$s.t. \sum_{j=1}^{n} \sum_{t=1}^{l} w_t |\delta_{tj}| = opti^*$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{l} w_t (f_{tj} + \delta_{tj})}{\sum_{i=1}^{m} v_i x_{ij}} = 1 \quad \forall j$$

$$\sum_{j=1}^{n} \delta_{tj} = 0 \quad \forall t \qquad (5)$$

$$f_{tj} + \delta_{tj} \geq 0 \quad \forall t, j$$

$$u_r, v_i, w_t \geq 0, \delta_{tj}\ free.$$

The objective function in Model (5) establishes the minimal and maximal efficiencies for DMU$_k$ subject to constraints in the same way as in Model (4), considering all possible equilibrium efficient frontiers. As stated in Theorem 1, Model (5) has a feasible solution always (see Appendix for a detailed proof).

Theorem 1: Model (5) is always feasible.

Model (5) is nonlinear. We first use Charnes–Cooper transformation (Charnes and Cooper, 1962) by setting $\frac{1}{\sum_{i=1}^{m} v_i x_{ik}} = d,\ v_i = d * v_i,\ u_r = d * u_r,\ w_t = d * w_t$ and transform it into

$$max/min(\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk})$$

$$s.t. \sum_{j=1}^{n}\sum_{t=1}^{l} w_t|\delta_{tj}| = d * opti^*$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{l} w_t(f_{tj} + \delta_{tj}) = 0 \quad \forall j$$

$$\sum_{j=1}^{n} \delta_{tj} = 0 \quad \forall t \qquad (6)$$

$$\sum_{i=1}^{m} v_i x_{ik} = 1$$

$$f_{tj} + \delta_{tj} \geq 0 \quad \forall t, j$$

$$u_r, v_i, w_t, d \geq 0, \delta_{tj} \text{ free.}$$

Second, we set $\delta'_{tj} = w_t \delta_{tj}$ and obtain the model

$$max/min(\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk})$$

$$s.t. \sum_{j=1}^{n}\sum_{t=1}^{l} |\delta'_{tj}| = d * opti^*$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{l} (w_t f_{tj} + \delta'_{tj}) = 0 \quad \forall j$$

$$\sum_{j=1}^{n} \delta'_{tj} = 0 \quad \forall t \qquad (7)$$

$$\sum_{i=1}^{m} v_i x_{ik} = 1$$

$$w_t f_{tj} + \delta'_{tj} \geq 0 \quad \forall t, j$$

$$u_r, v_i, w_t, d \geq 0, \delta'_{tj} \text{ free.}$$

Third, we set $a_{tj} = \frac{1}{2}(|\delta'_{tj}| + \delta'_{tj})$ and $b_{tj} = \frac{1}{2}(|\delta'_{tj}| - \delta'_{tj})$ (Si et al., 2013), and transform Model (7) into

$$max/min(\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk})$$

$$s.t. \sum_{j=1}^{n}\sum_{t=1}^{l}(a_{tj} + b_{tj}) = d * opti^*$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{l}(w_t f_{tj} + a_{tj} - b_{tj}) = 0 \quad \forall j$$

$$\sum_{j=1}^{n}(a_{tj} - b_{tj}) = 0 \quad \forall t \qquad (8)$$

$$\sum_{i=1}^{m} v_i x_{ik} = 1$$

$$w_t f_{tj} + a_{tj} - b_{tj} \geq 0 \quad \forall t, j$$

$$u_r, v_i, w_t, d, a_{tj}, b_{tj} \geq 0.$$

Denote the best and worst efficiencies of DMU$_k$ as e_k^{max} and e_k^{min}, respectively. Therefore, the efficiency of DMU$_k$ belongs to the interval $[e_k^{min}, e_k^{max}]$ when considering all feasible equilibrium efficient frontiers. In addition, we have the following theorem.

Theorem 2: The efficiency of DMU$_k$ assumes all values in the interval $[e_k^{min}, e_k^{max}]$.

By Theorem 2, the DMU$_k$'s efficiency is not unique and thus its efficiency rank relative DMUs may vary greatly. However, if the efficiency intervals of two DMUs do not overlap, then their relative ranks can be determined conclusively. For example, if DMU1's efficiency interval is [0.5, 0.8] and that of DMU2 is [0.9, 1.0], then DMU2 should have a higher rank than DMU1. However, in many cases the DMUs' efficiency intervals may overlap, which can be addressed as suggested in the following subsection. We also note that in the efficiency evaluations based on Model (5), the constraints are the same as in Model (4), and thus the DMUs are evaluated based all on equilibrium efficient frontiers.

3.2 Rank Intervals

This subsection proposes a series of models to calculate the rank interval of each DMU. Based on Salo and Punkka (2011), we define the sets

$R_k^{>}(u,v,w,\delta) = \{p \in \{1,\dots,n\} | E_p(u,v,w,\delta) > E_k(u,v,w,\delta)\},\ (u,v,w,\delta) \in S\}$

$R_k^{\geq}(u,v,w,\delta) = \{p \in \{1,\dots,n\} \backslash \{k\} | E_p(u,v,w,\delta) \geq E_k(u,v,w,\delta)\},\ (u,v,w,\delta) \in S\},$

where $R_k^{>}(u,v,w,\delta)$ contains other DMUs whose efficiency is strictly higher than that of DMU$_k$, while those whose efficiency ratios are at least as high that of DMU$_k$ are included in $R_k^{\geq}(u,v,w,\delta)$.

Correspondingly, the best and worst efficiency ranks are $r_k^{>}(u,v,w,\delta) = 1 + |R_k^{>}(u,v,w,\delta)|$ and $r_k^{\geq}(u,v,w,\delta) = 1 + |R_k^{\geq}(u,v,w,\delta)|$ (where $|R|$ denotes the number of elements in the set R). The difference between $r_k^{>}(u,v,w,\delta)$ and $r_k^{\geq}(u,v,w,\delta)$ is that if DMU$_k$ and DMU$_p$ are the two DMUs with the highest efficiency ratio along all DMUs, then the former rank them both as first, but the latter ranks them as second.

The bounds of ranking interval $[r_k^{min}, r_k^{max}]$for DMU$_k$ are

$$r_k^{min} = min_{(u,v,w,\delta)\in S}\, r_k^{>}(u,v,w,\delta) \text{ and } r_k^{max} = max_{(u,v,w,\delta)\in S}\, r_k^{\geq}(u,v,w,\delta).$$

Here, we propose the following model for calcultating the best (highest) ranking r_k^{min}:

$$r_k^{min} = min\,[\,1 + \sum_{p\neq k} z_p]$$

$$s.t. \sum_{j=1}^{n}\sum_{t=1}^{l} w_t \left|\delta_{tj}\right| = opti^*$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{l} w_t (f_{tj} + \delta_{tj})}{\sum_{i=1}^{m} v_i x_{ij}} = 1 \quad \forall j$$

$$\sum_{j=1}^{n} \delta_{tj} = 0 \quad \forall t$$

$$f_{tj} + \delta_{tj} \geq 0 \quad \forall t, j \qquad (9)$$

$$\frac{\sum_{r=1}^{s} u_r y_{rp} + \sum_{t=1}^{l} w_t f_{tp} - H z_p}{\sum_{i=1}^{m} v_i x_{ip}} \leq \frac{\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk}}{\sum_{i=1}^{m} v_i x_{ik}} \quad \forall p \in \{1, \dots, n\} \backslash \{k\}$$

$$z_p \in \{0,1\}, \quad \forall p \in \{1, \dots, n\} \backslash \{k\}$$

$$u_r, v_i, w_t, a_{tj}, b_{tj} \geq 0.$$

In Model (9), the first four sets of constraints are the same as in Model (4), ensuring that all possible equilibrium efficient frontiers are considered. The fifth set of constraints (where H is a large positive constant) treat DMU$_k$ as a benchmark and check whether DMU$_k$'s efficiency is strictly smaller than other DMUs' such that $E_k(u, v, w, \delta) < E_p(u, v, w, \delta), p \in \{1, \dots, n\} \backslash \{k\}$. If $E_k(u, v, w, \delta) < E_p(u, v, w, \delta)$, then $z_p = 1$, but otherwise Model (9) has no feasible solutions. In this case, each DMU$_p$ such that $E_k(u, v, w, \delta) < E_p(u, v, w, \delta), p \in \{1, \dots, n\} \backslash \{k\}$ has a higher rank than DMU$_k$. Therefore, the optimum value for Model (9) gives the best efficiency rank r_k^{min} for DMU$_k$.

Similarly, we calculate the worst (lowest) rank r_k^{max} of DMU$_k$ by solving the optimization problem

$$r_k^{max} = max \ [1 + \sum_{p \neq k} z_p]$$

$$s.t. \sum_{j=1}^{n} \sum_{t=1}^{l} w_t |\delta_{tj}| = opti^*$$

$$\frac{\sum_{r=1}^{s} u_r y_{rj} + \sum_{t=1}^{l} w_t (f_{tj} + \delta_{tj})}{\sum_{i=1}^{m} v_i x_{ij}} = 1 \quad \forall j$$

$$\sum_{j=1}^{n} \delta_{tj} = 0 \quad \forall t$$

$$f_{tj} + \delta_{tj} \geq 0 \quad \forall t, j \qquad (10)$$

$$\frac{\sum_{r=1}^{s} u_r y_{rp} + \sum_{t=1}^{l} w_t f_{tp} + H(1 - z_p)}{\sum_{i=1}^{m} v_i x_{ip}} \geq \frac{\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk}}{\sum_{i=1}^{m} v_i x_{ik}} \quad \forall p \in \{1, \dots, n\} \backslash \{k\}$$

$$z_p \in \{0,1\}, \quad \forall p \in \{1, \dots, n\} \backslash \{k\}$$

$$u_r, v_i, w_t, a_{tj}, b_{tj} \geq 0.$$

In Model (10), the fifth set of constraints again treats DMU$_k$ as a benchmark and checks whether DMU$_k$'s efficiency is equal to or smaller than other DMUs' such that $E_k(u, v, w, \delta) \leq E_p(u, v, w, \delta), p \in \{1, \dots, n\} \backslash \{k\}$. If $E_k(u, v, w, \delta) \leq E_p(u, v, w, \delta)$, then $z_p = 1$; otherwise, Model (10) does not have a feasible solution and any DMU$_p$ such that $E_k(u, v, w, \delta) \leq E_p(u, v, w, \delta), p \in \{1, \dots, n\} \backslash \{k\}$ should have a higher rank than DMU$_k$. Therefore, the solution to the Model (10) gives r_k^{max}, the worst efficiency rank for DMU$_k$.

Models (9) and (10) are nonlinear and thus optimal values can be hard to obtain. Salo and Punkka (2011) linearize their rank-based models in effect by setting $E_k(u, v, w, \delta) = 1$. However, we cannot set $E_k(u, v, w, \delta) = 1$, because this could violate the second set of constraints.

We therefore transform the two nonlinear models into linear ones as follows. Take Model (9) as an example and set $E_k(u, v, w, \delta) = h$. Based on Theorem 2, we know that h belongs to $[e_k^{min}, e_k^{max}]$. Thus, we transform the fifth set of constraint in model (9) into three parts as follows:

$$r_k^{min}(h) = min\,[\,1 + \sum_{p \neq k} z_p]$$

$$s.t. \sum_{j=1}^{n} \sum_{t=1}^{l} (a_{tj} + b_{tj}) - dC^* = 0$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{l} \left(w_t f_{tj} + a_{tj} - b_{tj}\right) = 0 \quad \forall j$$

$$\sum_{j=1}^{n} (a_{tj} - b_{tj}) = 0 \quad \forall t$$

$$w_t f_{tj} + a_{tj} - b_{tj} \geq 0 \quad \forall t, j \qquad (11)$$

$$\sum_{r=1}^{s} u_r y_{rp} - h \sum_{i=1}^{m} v_i x_{ip} + \sum_{t=1}^{l} w_t f_{tp} - H z_p \leq 0 \;\; \forall p \in \{1, \dots, n\} \backslash \{\text{k}\}$$

$$\sum_{i=1}^{m} v_i x_{ik} = 1$$

$$\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk} = h$$

$$z_p \in \{0,1\}, \quad \forall p \in \{1, \dots, n\} \backslash \{k\}$$

$$u_r, v_i, w_t, a_{tj}, b_{tj} \geq 0.$$

Model (11) can be regarded as a linear model with the parameter $h \in [e_k^{min}, e_k^{max}]$, and a heuristic search algorithm is proposed to calculate the best rank of DMU$_k$ $r_k^{min} = min\,\{r_k^{min}(h) |\, h \in [e_k^{min}, e_k^{max}]\}$ as follows:

Set $h = e_k^{max} - k \times \varepsilon$, where ε is the step size of the heuristic search algorithm and $k = 0,1,2, \dots, [k^{max}] + 1$. Here, $[k^{max}]$ is the largest integer value of $(e_k^{max} - e_k^{min})$ divided by the step size ε. In the iterative process of solving the model (11), we increase k from its initial value 0 to $[k^{max}] + 1$ with the step size ε and compute the corresponding $r_k^{min}(h)$ for each h. After check each k, we obtain the best rank $r_k^{min} = min\,\{r_k^{min}(h) |\, h \in [e_k^{min}, e_k^{max}]\}$.

Similarly, model (10) can be transformed to

$$r_k^{max}(h) = max\,[1 + \sum_{p \neq k} z_p]$$

$$s.t. \sum_{j=1}^{n} \sum_{t=1}^{l} (a_{tj} + b_{tj}) - dC^* = 0$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{l} \left(w_t f_{tj} + a_{tj} - b_{tj}\right) = 0 \quad \forall j$$

$$\sum_{j=1}^{n} (a_{tj} - b_{tj}) = 0 \quad \forall t$$

$$w_t f_{tj} + a_{tj} - b_{tj} \geq 0 \quad \forall t, j \qquad (12)$$

$$-\sum_{r=1}^{s} u_r y_{rp} + h \sum_{i=1}^{m} v_i x_{ip} - \sum_{t=1}^{l} w_t f_{tp} + H z_p \leq H \;\; \forall p \in \{1, \dots, n\} \backslash \{k\}$$

$$\sum_{i=1}^{m} v_i x_{ik} = 1$$

$$\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk} = h$$

$$z_p \in \{0,1\}, \quad \forall p \in \{1, \dots, n\} \backslash \{k\}$$

$$u_r, v_i, w_t, a_{tj}, b_{tj} \geq 0.$$

Model (12) is a linear model with the parameter $h \in [e_k^{min}, e_k^{max}]$, and the worst rank of DMU$_k$ is $r_k^{max} = max\,\{r_k^{max}(h) |\, h \in [e_k^{min}, e_k^{max}]\}$. Therefore, we can calculate the best (r_k^{min}) and worst (r_k^{max}) ranks for each DMU considering all feasible equilibrium efficient frontiers.

3.3 Efficiency Dominance

This subsection explores dominance relations among DMUs. We first extend the definition of efficiency dominance (Salo and Punkka, 2011) so that:

DEFINITION 1. DMU$_k$ dominates DMU$_g$ (denoted by $DMU_k \succ DMU_g$) if and only if

$$E_k(u, v, w, \delta) \geq E_g(u, v, w, \delta) \text{ for all } (u, v, w, \delta) \epsilon S$$

$$E_k(u, v, w, \delta) > E_g(u, v, w, \delta) \text{ for some } (u, v, w, \delta) \epsilon S.$$

This means that if DMU$_k$ dominates DMU$_g$, then the efficiency of DMU$_k$ must not be lower than that of DMU$_g$ for any feasible equilibrium efficient frontier and that it has to be strictly higher for some frontier. As shown by Salo and Punkka (2011), this dominance relation among DMUs is irreflexive, asymmetric and transitive.

We explore the dominance relation in Definition 1 by considering ratios between the efficiencies of DMU$_k$ and DMU$_g$ by defining

$$D_{k,g}(u, v, w, \delta) = \frac{E_k(u, v, w, \delta)}{E_g(u, v, w, \delta)}, \qquad (u, v, w, \delta) \in S \quad (13).$$

Because there are many feasible equilibrium efficient frontiers in S, we can calculate the maximum and minimum values of $D_{k,g}(u, v, w, \delta)$, denoted by $\overline{D}_{k,g}$ and $\underline{D}_{k,g}$, respectively, as follows:

$$max/min\,[\sum_{r=1}^{s} u_r y_{rk} + \sum_{t=1}^{l} w_t f_{tk}]/h$$

$$s.t. \sum_{j=1}^{n} \sum_{t=1}^{l} (a_{tj} + b_{tj}) - dC^* = 0$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{l} \left(w_t f_{tj} + a_{tj} - b_{tj}\right) = 0 \quad \forall j$$

$$\sum_{j=1}^{n} (a_{tj} - b_{tj}) = 0 \quad \forall t$$

$$w_t f_{tj} + a_{tj} - b_{tj} \geq 0 \quad \forall t, j \qquad (14)$$

$$\sum_{i=1}^{m} v_i x_{ik} = 1$$

$$\sum_{r=1}^{s} u_r y_{rg} - h \sum_{i=1}^{m} v_i x_{ig} + \sum_{t=1}^{l} w_t f_{tg} = 0$$

$$u_r, v_i, w_t, a_{tj}, b_{tj} \geq 0.$$

In Model (14), DMU_g serves as a benchmark whose efficiency is set to $E_g(u, v, w, \delta) = h$, $h \in [e_g^{min}, e_g^{max}]$. Thus, Model (14) can be treated as a linear one with a parameter h so that we can use an similar algorithm similar to that for Model (11) to calculate $\overline{D}_{k,g}$ and $\underline{D}_{k,g}$, respectively. For instance, if $\overline{D}_{k,g} = 1.33$ and $\underline{D}_{k,g} = 1.18$, the efficiency of DMU_k can be at most 33% but not less than 18% greater than that of DMU_g. Specifically, DMU_k dominates DMU_g if the minimum value $\underline{D}_{k,g} > 1$ or if $\underline{D}_{k,g} = 1$ and $\overline{D}_{k,g} > 1$; otherwise, DMU_k does not dominate DMU_g. Thus, for example, if , there is no dominance for the two DMUs.

4. Dataset from Yang et al. (2011)

This section illustrates the above models and compares them with CCR and GEEFDEA methods. We first revisit the data set in Yang et al. (2011) on 6 DMUs which produce a single fixed-sum output (see Table 1).

Table 1 Dataset from Yang et al. (2011).

DMU	A	B	C	D	E	F
Input	1	1	1	1	1	1
Unfixed-sum output 1	3	1	3	5	2	5
Fixed-sum output 2	3	4	1	1	2	2

Table 2 shows the results based on CCR model, GEEFCCR model and the models proposed in this paper. The second column shows that DMUs A, B, D and F are efficient based on the CCR model and they have the same rank. The third column shows the results for GEEFCCR models, indicating that there are some efficient DMUs whose efficiencies exceed 1 with the rank ordering $F > A > D > B$. However, the DMUs' efficiency scores and ranks based on GEEFCCR models are calculated by choosing only one feasible equilibrium efficient frontier, even if there are many feasible equilibrium efficient frontiers so that the DMUs' efficiency scores and ranks may vary greatly depending on which efficient frontier is chosen (Fang, 2016; Zhu et al., 2017).

Table 2 Results based on three models.

DMU	CCR model		GEEFCCR model		Proposed models in this paper		
	score	rank	score	rank	eff interval	ranking interval	dominated by
A	1	1	1.1429	2	[1.1077, 1.3846]	[2,3]	—
B	1	1	1	4	[0.8769, 1.8462]	[1,4]	—
C	0.6	6	0.7302	6	[0.4615, 0.7692]	[5,6]	A,B,D,F
D	1	1	1.0794	3	[0.4615, 1.1692]	[2,6]	F
E	0.6667	5	0.7619	5	[0.7385, 0.9231]	[3,6]	A,B,F
F	1	1	1.2857	1	[0.9231, 1.3385]	[1,4]	—

The last column shows the efficiency intervals, rank intervals and dominance relations based on the proposed models by considering all feasible equilibrium efficient frontiers. First, we see that each DMU's efficiency based on the GEFECCR model takes a value which belongs to efficiency interval based on Model (5). For example, based on GEFECCR model, the efficiency for DMU_A is 1.1429, which is in the efficiency interval [1.1077, 1.3846] given by Model (5).

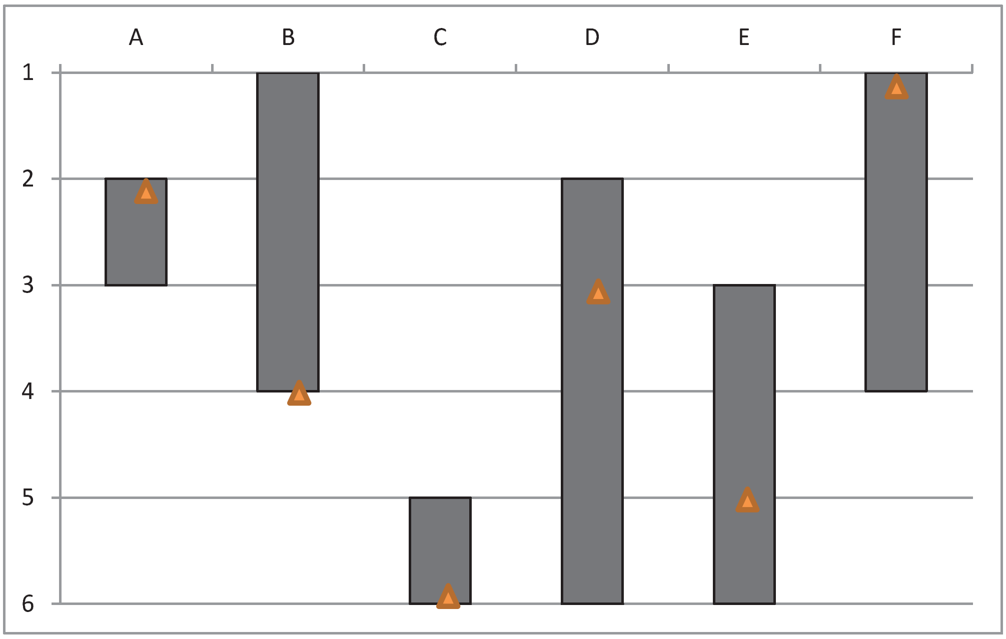

Figure1. Rank comparison based on the GEEFCCR method and the proposed method.

Second, we note that the rank of each DMU under the GEEFCCR method is a special case of the models proposed in this paper. For example, in the GEEFCCR method the rank of DMU_A is 2, which is in the corresponding rank interval [2, 3] for DMU_A (see Figure 1 for details). The orange triangle denotes the rank order of each DMU based on the GEEFCCR method, while the length of cylindrical shape shows the rank interval for each DMU based on our models.

Third, the proposed models imply the dominance relations in Figure 2. Specifically DMU_F dominates DMU_D, while DMU_D and DMU_F both are CCR-efficient. Still, DMU_F does not dominate DMU_A and DMU_B, even though it has the highest ranking under the GEEFCCR method.

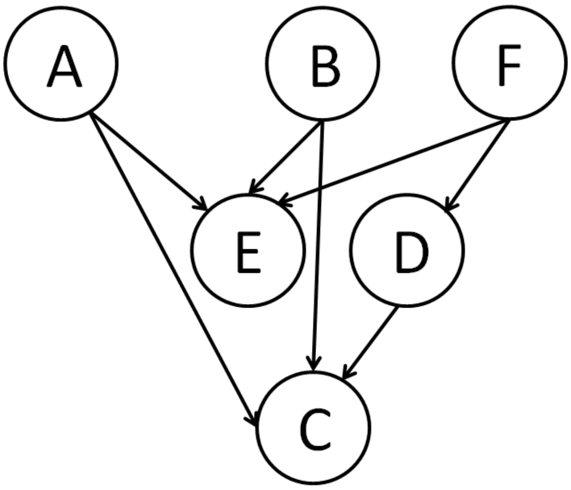

Figure 2. Efficiency dominance relations among DMUs.

An interesting finding is that DMU_C is dominated by DMU_B based on the approach we have proposed even if there is no such dominance relation based on the approach proposed by Salo and Punkka (2011). Specifically, the outputs of DMU_B and DMU_C are (1,4) and (3,1), respectively, and consequently neither DMU dominates the other based on the approach proposed by Salo and Punkka (2011). Moreover, we find that DMU_E is dominated by DMU_B based on our approach. Therefore, our approach can establish additional dominance relations among the DMUs based on the efficiency evaluation with fixed-sum outputs.

Compared to GEEFDEA methods, the proposed efficiency measures help evaluate DMUs with fixed-sum outputs more objectively and comprehensively. In addition to efficiency values, rank intervals and dominance relations of DMUs provide more information about the performance of DMUs in the context of fixed-sum outputs.

5. Apply to Appliance Industry Companies

This section applies the proposed approach to the dataset on the appliance industry considered in Yang et al. (2014). This dataset (see Table 3) has 18 appliance industry companies with two inputs (total assets (million yuan) and number of employees) and two fixed-sum outputs (profits (million yuan) and market share). The total profit of these companies is a constant and the total market share is 100%.

Table 3 Dataset of appliance industry companies of China in 2012

Company	Total assets	Employees	Profits	Market share
Guangdong Midea	59550	66497	3699	12.9267
Qingdao Haier	39723	59814	2690	10.2271
GOME	37227	59624	1840	8.7351
Zhuhai Gree	85212	72671	5237	11.5951
Sichuan Changhong	51651	60398	406	7.2199
Suning Appliance	59786	7751	4821	13.0351
TCL Corporation	74014	56190	1013	8.4459
Hisense Electric	16145	15776	1689	3.2660
Skyworth Digital	17762	26000	1039	2.9786
Konka Group	16906	19724	25	2.2516
Haier Electronics	14294	18406	1406	6.9303
Hisense Kelon	7635	31010	227	2.5669
TCL Multimedia	19564	26275	367	3.7723
Wuxi Little Swan	9145	1073	453	1.5239
Hefei Meiling	7603	4048	107	1.2501
Shanghai Highly	7116	3166	174	1.1354
Zhejiang Supor	4392	11371	476	0.9893
Zhejiang Chint	8357	11286	824	1.1507

For convenience, we refer to these 18 companies as *F1-F18* in the above order. Efficiency results for these companies based on the CCR model, GEEFCCR models and our proposed models are in Table 4.

Table 4 Evaluation results based on three different approaches

DMU	CCR model		GEEFCCR model		Proposed models in this paper		
	score	rank	score	rank	Efficiency interval	Rank interval	dominated by
F1	0.6172	9	1.1387	5	[1.0713, 1.2569]	[4, 9]	6,11
F2	0.6676	8	1.2342	4	[0.9355, 1.3802]	[2, 10]	11
F3	0.4995	14	1.1015	6	[0.6419, 1.2579]	[4, 13]	2,11
F4	0.6134	10	0.7625	14	[0.7295, 1.4990]	[5, 16]	6,8,11,14
F5	0.3055	17	0.7240	15	[0.1398, 0.7493]	[14, 17]	1,2,3,6,8,11,13,14,15,16
F6	1.0000	1	1.9770	2	[1.1688, 12.9379]	[1, 6]	—
F7	0.3143	16	0.6553	18	[0.2769, 0.8283]	[10, 18]	1,2,3,4,6,8,11,14,15,16
F8	1.0000	1	1.0983	7	[1.0844, 2.2270]	[1, 8]	6
F9	0.5649	11	0.8114	13	[0.6313, 1.1837]	[8, 15]	1,2,6,8,11
F10	0.2915	18	0.6903	16	[0.0264, 0.7140]	[15, 18]	1,2,3,4,5,6,8,9,11,13,14,15,16
F11	1.0000	1	2.4422	1	[1.5889, 2.5991]	[1, 4]	—
F12	0.6934	7	1.0294	8	[0.1523, 1.8023]	[2, 18]	11
F13	0.3977	15	0.9589	11	[0.2905, 1.0337]	[9, 15]	1,2,3,6,8,11
F14	0.8445	6	1.2910	3	[0.8933, 8.7818]	[2, 11]	6
F15	0.5290	13	1.0047	9	[0.2848, 1.7018]	[5, 15]	6,11,14
F16	0.5489	12	0.9996	10	[0.4948, 1.9763]	[4, 14]	6,11,14
F17	1.0000	1	0.8710	12	[0.4795, 2.1930]	[1, 18]	—
F18	0.9347	5	0.6836	17	[0.5619, 1.9952]	[3, 16]	6,8,11

The second column shows that the four DMUs *F6*, *F8*, *F11* and *F17* are CCR-efficient while others are inefficient. The third column shows that *F11* is the best in the CEEFCCR model. The last column shows that *F6*, *F11* and *F17* are not dominated by any other DMUs, but the CCR-efficient *F8* is dominated by *F6*. Interestingly, all three approaches regard *F10* as the worst DMU, which is dominated by all others except *F12*.

In addition, we find that the efficiency of each DMU based on the GEEFCCR approach is a scalar value within the interval which is defined by the minimal or maximal efficiencies based on the proposed approach in this paper. The analogous result also holds for the rank of each DMU based on the two approaches. In particular, these results can be attributed to the fact that GEEFCCR approach evaluates DMUs just based on just one feasible equilibrium efficient frontier, disregarding other equally feasible equilibrium efficient frontiers.

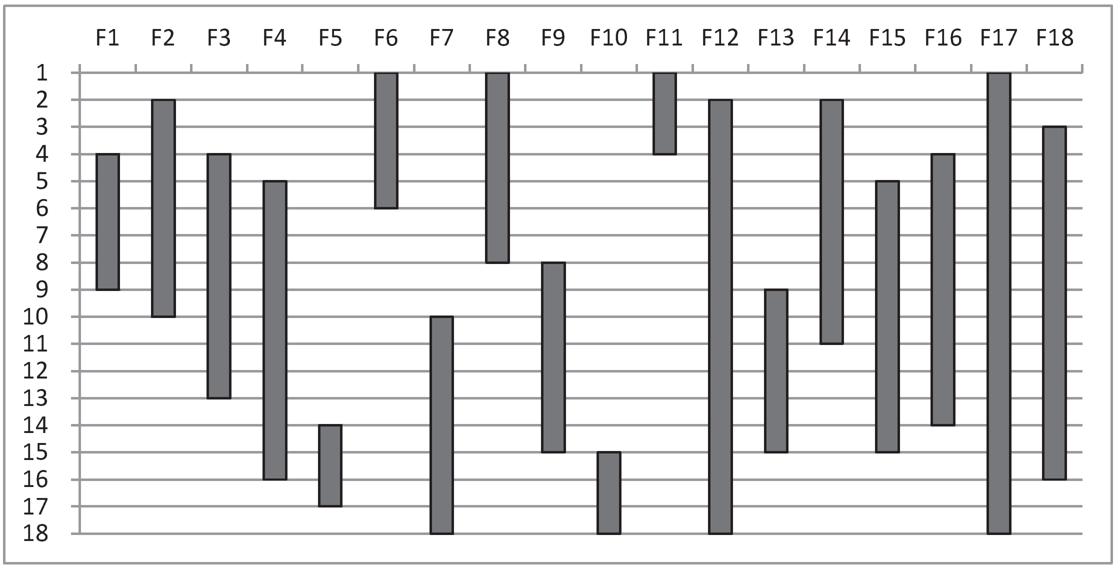

Figure 3 Rank intervals for 18 companies.

The rank intervals and dominance relations show that there are no domination relations among *F6*, *F11* and *F17*, indicating that DMUs may perform quite differently for different equilibrium efficient frontiers. From Figure 3, we see that the rank of each firm in the GEEFCCR method is included in the corresponding ranking interval calculated based on all feasible equilibrium efficient frontiers. However, the nondominated *F17* can have the worst rank 18 when some equilibrium efficient frontiers are chosen, even though it is efficient in the the CCR model. In constrast, the rank of *F11* can be the best one but never worse than 4, and thus it is more robust than *F6* and *F17* based on efficiency ranks, although *F6*, *F11* and *F17* are all nondominated.

Figure 3 shows that *F2*, *F12*, *F14* are CCR-inefficient, but they could be the second best DMUs for some feasible equilibrium efficient frontiers. Another result is that *F1* ranks 9^{th} based on the CCR model, while it can assume ranks in the range from the 4^{th} to the 9^{th} based on the proposed approach in this paper.

6. Conclusion and Future Research

In this paper, we have developed several models to calculate the efficiency and rank intervals as well as dominance relations for DMUs which produce fixed-sum outputs. In contrast to previous approaches, these results are based on the consideration of all feasible equilibrium efficient frontiers instead of a single efficient frontier. We have compared our models with previous approaches, evaluating them by using the dataset in Yang et al. (2011) to show that they offer more informative results. We have also applied them to evaluate the performance of appliance industry companies of China in 2012 more comprehensively than what would be permitted by CCR and GEEFCCR models, providing evidence on the usefulness of our methodological research.

However, we have not considered undesirable fixed-sum outputs such as hazardous waste and noise. Therefore, the development of models for undesirable fixed-sum outputs calls for further research. Another direction for continued research direction is that of extending the proposed models to situations where the assumption of constant returns to scale needs to be replaced by variable returns to scale.

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Appendix A.

Model (2) is a nonlinear model. Yang et al. (2015) transform it to be a linear one by two steps. In the first step, they set $\delta'_{tj} = w_t \delta_{tj}$ and obtain the model (A1):

$$\mathrm{Min} \sum_{j=1}^{n} \sum_{t=1}^{l} |\delta'_{tj}|$$

$$s.t. \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{l} (w_t f_{tj} + \delta'_{tj}) = 0 \quad \forall i, r, t, j$$

$$\sum_{j=1}^{n} \delta'_{tj} = 0 \quad \forall t \qquad \text{(A1)}$$

$$\sum_{i=1}^{m} v_i x_{ij} \geq C \, \forall j$$

$$w_t f_{tj} + \delta'_{tj} \geq 0, \quad \forall t, j$$

$$u_r, v_i, w_t \geq 0, \delta'_{tj} \text{ free,}$$

where C in the model (A1) is a positive constant that guarantees the denominator $\sum_{i=1}^{m} v_i x_{ij}$ in the model (2) is positive and thus guarantees that the objective function of model (A1) must be positive; otherwise, this function is zero.

In the second step, they set $a_{tj} = \frac{1}{2}(|\delta'_{tj}| + \delta'_{tj})$ and $b_{tj} = \frac{1}{2}(|\delta'_{tj}| - \delta'_{tj})$ (see Si et al., 2013) so that Model (A2) is transformed to

$$Min \sum_{j=1}^{n} \sum_{t=1}^{l} (a_{tj} + b_{tj})$$

$$s.t. \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \sum_{t=1}^{l} (w_t f_{tj} + a_{tj} - b_{tj}) = 0 \quad \forall i, r, t, j$$

$$\sum_{j=1}^{n} (a_{tj} - b_{tj}) = 0 \quad \forall t \qquad (A2)$$

$$\sum_{i=1}^{m} v_i x_{ij} \geq C \, \forall j$$

$$w_t f_{tj} + a_{tj} - b_{tj} \geq 0 \quad \forall t, j$$

$$u_r, v_i, w_t, a_{tj}, b_{tj} \geq 0.$$

Appendix B. Proof of Theorem 1

By Yang et al. (2015), we know that Model (2) is always feasible, and thus the optimum $\left(v_i^*, u_r^*, w_t^*, \delta_{tj}^*\right)(\forall i, r, t, j)$ exists. At this optimum, the objective function obtains its maximum value and all constraints in Model (2) hold so that

$$\sum_{j=1}^{n}\sum_{t=1}^{l} w_t^*\left|\delta_{tj}^*\right| = opti^*$$

$$\frac{\sum_{r=1}^{s} u_r^* y_{rj} + \sum_{t=1}^{l} w_t^*(f_{tj}^* + \delta_{tj}^*)}{\sum_{i=1}^{m} v_i^* x_{ij}} = 1 \quad \forall j$$

$$\sum_{j=1}^{n} \delta_{tj}^* = 0 \quad \forall t \qquad (2')$$

$$f_{tj}^* + \delta_{tj}^* \geq 0 \quad \forall t, j$$

$$u_r^*, v_i^*, w_t^* \geq 0, \delta_{tj}^*\ free.$$

A comparison of Models (2') and (5) shows that these two models have the same constraints and $\left(v_i^*, u_r^*, w_t^*, \delta_{tj}^*\right)(\forall i, r, t, j)$ also satisfies all the constraints of Model (5). Therefore, $\left(v_i^*, u_r^*, w_t^*, \delta_{tj}^*\right)(\forall i, r, t, j)$ is a feasible solution of Model (5), too.

Appendix C. Proof of Theorem 2

Theorem 2: Each DMU$_k$'s efficiency is continuous in an interval$[e_k^{min}, e_k^{max}]$.

For any efficiency value of DMU$_k$ on all feasible equilibrium efficient frontiers $e = e_0$, $\forall e_0 \in [e_k^{min}, e_k^{max}]$, the feasible solution to model (6) is denoted as $\left(v_i^*, u_r^*, w_t^*, \delta_{tj}^*, \forall i, r, t, j\right)$. Here, $e_0 = \sum_{r=1}^{s} u_r^* y_{rk} + \sum_{t=1}^{l} w_t^* f_{tk}$.

This theorem is proven in three steps:

(i) $\boldsymbol{e_0 = e_k^{min}}$**.**

Let $e' = e_0 + \Delta e, \Delta e > 0, e' \in [e_k^{min}, e_k^{max}]$, $u_r' = u_r^*(e_0 + \Delta e)/e_0$, $w_t' = w_t^*(e_0 + \Delta e)/e_0$, $\delta_{tj}' = \delta_{tj}^* e_0/(e_0 + \Delta e)$, then $(v_i^*, u_r', w_t', \delta_{tj}', \forall i, r, t, j)$ is the feasible solution to model (6) for $e = e'$ when $\Delta e \to 0$, because $e' = \sum_{r=1}^{s} u_r' y_{rk} + \sum_{t=1}^{l} w_t' f_{tk}$ and the solution satisfies all the constraints of model (6), such as:

$$\sum_{j=1}^{n}\sum_{t=1}^{l} w_t'\left|\delta_{tj}'\right| = \sum_{j=1}^{n}\sum_{t=1}^{l} w_t^*\left|\delta_{tj}^*\right| = d * opti^*$$

$$\sum_{j=1}^{n} \delta_{tj}' = \sum_{j=1}^{n} \delta_{tj}^* * \frac{e_0}{e_0+\Delta e} = 0 \quad \forall t$$

$$\sum_{i=1}^{m} v_i^* x_{ik} = 1$$

$$f_{tj} + \delta_{tj}' = \ f_{tj} + \delta_{tj}^* \ * \frac{e_0}{e_0+\Delta e} \xrightarrow{\Delta e \to 0} f_{tj} + \delta_{tj}^* \geq 0 \quad \forall t, j$$

$$\sum_{r=1}^{s} u_r' y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w_t'(f_{tj} + \delta_{tj}')$$

$$= \ \sum_{r=1}^{s} u_r^* y_{rk}\left(1+\frac{\Delta e}{e_0}\right) - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w_t^* f_{tj}\left(1+\frac{\Delta e}{e_0}\right) + \sum_{t=1}^{l} w_t^* \delta_{tj}^*$$

$$= \ (\sum_{r=1}^{s} u_r^* y_{rk} + \sum_{t=1}^{l} w_t^* f_{tj}) * \frac{\Delta e}{e_0} + \sum_{r=1}^{s} u_r^* y_{rk} - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w_t^*(f_{tj} + \delta_{tj}^*)$$

$$= (\sum_{r=1}^{s} u_r^* y_{rk} + \sum_{t=1}^{l} w_t^* f_{tj}) * \frac{\Delta e}{e_0} \xrightarrow{\Delta e \to 0} 0 \quad \forall j$$

Thus, each DMU$_k$'s efficiency on all feasible equilibrium efficient frontiers is right continuous at the point e_k^{min}.

(ii) $\boldsymbol{e_0 = e_k^{max}}$

Let $e' = e_0 - \Delta e, \Delta e > 0, e' \in [e_k^{min}, e_k^{max}]$, $u_r' = u_r^*(e_0 - \Delta e)/e_0$, $w_t' = w_t^*(e_0 - \Delta e)/e_0$, $\delta_{tj}' = \delta_{tj}^* e_0/(e_0 - \Delta e)$, then $(v_i^*, u_r', w_t', \delta_{tj}'), \forall i, r, t, j$ is the feasible solution to model (6) for $e = e'$ when $\Delta e \to 0$, because $e' = \sum_{r=1}^{s} u_r' y_{rk} + \sum_{t=1}^{l} w_t' f_{tk}$ and the solution satisfies all the constraints of Model (6) such as

$$\sum_{j=1}^{n}\sum_{t=1}^{l} w_t'\left|\delta_{tj}'\right| = \sum_{j=1}^{n}\sum_{t=1}^{l} w_t^*\left|\delta_{tj}^*\right| = d * opti^*$$

$$\sum_{j=1}^{n} \delta_{tj}' = \sum_{j=1}^{n} \delta_{tj}^* * \frac{e_0}{e_0-\Delta e} = 0 \quad \forall t$$

$$\sum_{i=1}^{m} v_i^* x_{ik} = 1$$

$$f_{tj} + \delta'_{tj} = \ f_{tj} + \delta^*_{tj} \ * \frac{e_0}{e_0 - \Delta e} \xrightarrow{\Delta e \to 0} f_{tj} + \delta^*_{tj} \geq 0 \quad \forall t, j$$

$$\sum_{r=1}^{s} u'_r y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w'_t (f_{tj} + \delta'_{tj})$$

$$= \ \sum_{r=1}^{s} u^*_r y_{rk} \left(1 - \frac{\Delta e}{e_0}\right) - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w^*_t f_{tj} \left(1 - \frac{\Delta e}{e_0}\right) + \sum_{t=1}^{l} w^*_t \delta^*_{tj}$$

$$= -(\sum_{r=1}^{s} u^*_r y_{rk} + \sum_{t=1}^{l} w^*_t f_{tj}) * \frac{\Delta e}{e_0} \xrightarrow{\Delta e \to 0} 0 \quad \forall j$$

Thus, each DMU$_k$'s efficiency on all feasible equilibrium efficient frontiers is left continuous at the point e_k^{max}.

(iii) $\boldsymbol{e_k^{min} < e_0 < e_k^{max}}$

First, let $e' = e_0 + \Delta e, \Delta e > 0, e' \in [e_k^{min}, e_k^{max}]$, $u'_r = u^*_r (e_0 + \Delta e)/e_0$, $w'_t = \frac{w^*_t (e_0 + \Delta e)}{e_0}$, $\delta'_{tj} = \delta^*_{tj} e_0/(e_0 + \Delta e)$, then $(v_i^*, u'_r, w'_t, \delta'_{tj}), \forall i, r, t, j$ is the feasible solution to model (6) for $e = e'$ when $\Delta e \to 0$, because $e' = \sum_{r=1}^{s} u'_r y_{rk} + \sum_{t=1}^{l} w'_t f_{tk}$ and the solution satisfies all the constraints of model (6) so that

$$\sum_{j=1}^{n} \sum_{t=1}^{l} w'_t |\delta'_{tj}| = \sum_{j=1}^{n} \sum_{t=1}^{l} w^*_t |\delta^*_{tj}| = d * opti^*$$

$$\sum_{j=1}^{n} \delta'_{tj} = \sum_{j=1}^{n} \delta^*_{tj} * \frac{e_0}{e_0 + \Delta e} = 0 \quad \forall t$$

$$\sum_{i=1}^{m} v_i^* x_{ik} = 1$$

$$f_{tj} + \delta'_{tj} = \ f_{tj} + \delta^*_{tj} \ * \frac{e_0}{e_0 + \Delta e} \xrightarrow{\Delta e \to 0} f_{tj} + \delta^*_{tj} \geq 0 \quad \forall t, j$$

$$\sum_{r=1}^{s} u'_r y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w'_t (f_{tj} + \delta'_{tj})$$

$$= \ \sum_{r=1}^{s} u^*_r y_{rk} \left(1 + \frac{\Delta e}{e_0}\right) - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w^*_t f_{tj} \left(1 + \frac{\Delta e}{e_0}\right) + \sum_{t=1}^{l} w^*_t \delta^*_{tj}$$

$$= (\sum_{r=1}^{s} u_r^* y_{rk} + \sum_{t=1}^{l} w_t^* f_{tj}) * \frac{\Delta e}{e_0} + \sum_{r=1}^{s} u_r^* y_{rk} - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w_t^* (f_{tj} + \delta_{tj}^*)$$

$$= (\sum_{r=1}^{s} u_r^* y_{rk} + \sum_{t=1}^{l} w_t^* f_{tj}) * \frac{\Delta e}{e_0} \xrightarrow{\Delta e \to 0} 0 \quad \forall j$$

Second, let $e' = e_0 - \Delta e, \Delta e > 0, e' \in [e_k^{min}, e_k^{max}]$, $u_r' = u_r^*(e_0 - \Delta e)/e_0$, $w_t' = \frac{w_t^*(e_0 - \Delta e)}{e_0}$, $\delta_{tj}' = \delta_{tj}^* e_0/(e_0 - \Delta e)$, then $(v_i^*, u_r', w_t', \delta_{tj}'), \forall i, r, t, j$ is the feasible solution to model (6) for $e = e'$ when $\Delta e \to 0$, because $e' = \sum_{r=1}^{s} u_r' y_{rk} + \sum_{t=1}^{l} w_t' f_{tk}$ and the solution satisfies all the constraints of Model (6) such as

$$\sum_{j=1}^{n} \sum_{t=1}^{l} w_t' |\delta_{tj}'| = \sum_{j=1}^{n} \sum_{t=1}^{l} w_t^* |\delta_{tj}^*| = d * opti^*$$

$$\sum_{j=1}^{n} \delta_{tj}' = \sum_{j=1}^{n} \delta_{tj}^* * \frac{e_0}{e_0 - \Delta e} = 0 \quad \forall t$$

$$\sum_{i=1}^{m} v_i^* x_{ik} = 1$$

$$f_{tj} + \delta_{tj}' = f_{tj} + \delta_{tj}^* * \frac{e_0}{e_0 - \Delta e} \xrightarrow{\Delta e \to 0} f_{tj} + \delta_{tj}^* \geq 0 \quad \forall t, j$$

$$\sum_{r=1}^{s} u_r' y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w_t' (f_{tj} + \delta_{tj}')$$

$$= \sum_{r=1}^{s} u_r^* y_{rk} \left(1 - \frac{\Delta e}{e_0}\right) - \sum_{i=1}^{m} v_i^* x_{ij} + \sum_{t=1}^{l} w_t^* f_{tj} \left(1 - \frac{\Delta e}{e_0}\right) + \sum_{t=1}^{l} w_t^* \delta_{tj}^*$$

$$= -(\sum_{r=1}^{s} u_r^* y_{rk} + \sum_{t=1}^{l} w_t^* f_{tj}) * \frac{\Delta e}{e_0} \xrightarrow{\Delta e \to 0} 0 \quad \forall j$$

Thus, each DMU$_k$'s efficiency on all feasible equilibrium efficient frontiers is continuous at every point in the interval $[e_k^{min}, e_k^{max}]$.

In summary, Each DMU$_k$'s efficiency is continuous in an interval $[e_k^{min}, e_k^{max}]$, completing the proof of Theorem 2.